A SAT-Based Approach
for Solving the Minimal S5-Satisfiability Problem

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Introduction: SAT solving

For many \textbf{NP} problems, here the current most efficient technique:

\begin{center}
\begin{tikzpicture}
  \node [fill=lightgray!50, rounded rectangle] (phi) at (0,0) {Problem $\phi$};
  \node [fill=red!30, rounded rectangle] (psi) at (0,-1.5) {SAT problem $\psi$};
  \node [fill=red!30, rounded rectangle] (check) at (0,-3) {check($\psi$)};
  \node [fill=lightgray!50, rounded rectangle] (solutionphi) at (2.5,0) {Solution $S_\phi \models \phi$};
  \node [fill=red!30, rounded rectangle] (solutionpsi) at (2.5,-1.5) {Solution $S_\psi \models \psi$};

  \draw [->, >=stealth] (phi) -- (psi) node [midway, above] {encoding};
  \draw [->, >=stealth] (psi) -- (check) node [midway, above] {solving};
  \draw [->, >=stealth] (check) -- (solutionphi) node [midway, above] {interpretation};
  \draw [->, >=stealth] (solutionphi) -- (solutionpsi) node [midway, above] {encoding};
\end{tikzpicture}
\end{center}
Introduction: S5-Satisfiability Problem as SAT

S52SAT solving

Problem $\phi$
- encoding
- Modal Logic
  S5 problem $\mu$
  - translate
  SAT problem $\psi$
  - solving
  check($\psi$)

Solution $S_{\phi} \models \phi$

Solution $S_{\mu} \models \mu$

Solution $S_{\psi} \models \psi$
Introduction: Why we want to minimize?

Motivations

- Modal Logic S5 is more expressive than propositional logic
- They have the same complexity (\(NP\)-complete)
- Modal logic S5 models are large (roughly one variable assignment per world)
- For some polynomial reduction to ML-S5, we may want a small model

Question: How to obtain the smallest model in number of worlds for modal logic S5?
Preliminaries: Modal Logic S5

Modal Logic S5

- $\phi ::= \top \mid p \mid \neg \phi \mid \phi \land \phi \mid \phi \lor \phi \mid \Box \phi \mid \Diamond \phi$
- $\Box \phi$ means $\phi$ is necessarily true
- $\Diamond \phi$ means $\phi$ is possibly true
- There exists a lot of different modal logics
- Most of them have their satisfiability in $\text{PSPACE}$

Satisfiability in Modal Logic S5 is $\textbf{NP}$-Complete [Lad77]
Preliminaries: S5-Structure

- $\mathcal{P}$ finite non-empty set of propositional variables

S5-Structure $\mathcal{K}$ [Kri63]

$M = \langle W, R, V \rangle$ with:

- $W$, a non-empty set of possible worlds
- $R$, a binary relation on $W$ (which is total: $\forall w. \forall v. (w, v) \in R$)
- $V$, a function that associate to each $p \in \mathcal{P}$, the set of possible worlds where $p$ is true

Pointed S5-Structure: $\langle \mathcal{K}, w \rangle$

- $\mathcal{K}$: S5-Structure
- $w$: a possible world in $W$
**Preliminaries: S5-Satisfiability**

**Definition (Satisfaction Relation)**

The relation $\models$ between S5-Structures and formulae is recursively defined as follows:

- $\langle K, w \rangle \models p$ iff $w \in V(p)$
- $\langle K, w \rangle \models \neg \phi$ iff $\langle K, w \rangle \not\models \phi$
- $\langle K, w \rangle \models \phi_1 \land \phi_2$ iff $\langle K, w \rangle \models \phi_1$ and $\langle K, w \rangle \models \phi_2$
- $\langle K, w \rangle \models \phi_1 \lor \phi_2$ iff $\langle K, w \rangle \models \phi_1$ or $\langle K, w \rangle \models \phi_2$
- $\langle K, w \rangle \models \Box \phi$ iff $(w, w') \in R$ implies $\langle K, w' \rangle \models \phi$
- $\langle K, w \rangle \models \Diamond \phi$ iff $(w, w') \in R$ and $\langle K, w' \rangle \models \phi$

A $K$ that satisfied a formula $\phi$ will be called “model of $\phi$”
Preliminaries: S5-Satisfiability

\[ \phi_1 = \Box (\bullet) \]

\[ \phi_2 = \Box \Diamond (\bullet) \]

\[ \phi_3 = (\bullet \lor \bullet) \]

\[ \phi_4 = \Diamond (\bullet \land \Box \neg \bullet) \]
Preliminaries: Upper-bound on $\mathcal{K}$

$\phi$ is considered in NNF (negation only on variables)

**Definition (Diamond-Degree)**

\[
\begin{align*}
\text{dd}(\top) &= \text{dd}(\neg \top) = 0 \\
\text{dd}(p) &= \text{dd}(\neg p) = 0 \\
\text{dd}(\psi \land \chi) &= \text{dd}(\psi) + \text{dd}(\chi) \\
\text{dd}(\psi \lor \chi) &= \max(\text{dd}(\psi), \text{dd}(\chi)) \\
\text{dd}(\Box \psi) &= \text{dd}(\psi) \\
\text{dd}(\Diamond \psi) &= 1 + \text{dd}(\psi)
\end{align*}
\]

For any $\mathcal{K}$ satisfying a formula $\phi$ in S5, $|\mathcal{K}| \leq \text{dd}(\phi) + 1$
From Modal Logic to SAT

\[
\begin{align*}
\text{tr}(\phi, n) &= \text{tr}'(\phi, 1, n) \\
\text{tr}'(p, i, n) &= p_i \quad \text{tr}'(\neg p, i, n) = \neg p_i \\
\text{tr}'(\psi \land \chi, i, n) &= \text{tr}'(\psi, i, n) \land \text{tr}'(\chi, i, n) \\
\text{tr}'(\psi \lor \chi, i, n) &= \text{tr}'(\psi, i, n) \lor \text{tr}'(\chi, i, n) \\
\text{tr}'(\Box \psi, i, n) &= \bigwedge_{j=1}^{n} (\text{tr}'(\psi, j, n)) \\
\text{tr}'(\Diamond \psi, i, n) &= \bigvee_{j=1}^{n} (\text{tr}'(\psi, j, n))
\end{align*}
\]

\[S52SAT = \text{tr}(\text{nnf}(\phi), \text{dd}(\phi) + 1)\]
Preliminaries: Goal

What we want

- We know how to find a $\mathcal{K}$ that satisfies a formula
- We know that an upper-bound for $|\mathcal{K}|$ is $\text{dd}(\phi) + 1$ \cite{CLL+17}
- The solver S52SAT translates $\phi$ into SAT to find it \cite{CLL+17}
- S52SAT always find a model of size $\text{dd}(\phi) + 1$ by construction

How to find the smallest $\mathcal{K}$ that satisfied a formula?
What we want

- We know how to find a $\mathcal{K}$ that satisfies a formula
- We know that an upper-bound for $|\mathcal{K}|$ is $dd(\phi) + 1$ \cite{CLL+17}
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How to find the smallest $\mathcal{K}$ that satisfied a formula?

Many different techniques that we will discuss here:

- MaxSAT solvers
- MCS Extraction
- Unsatisfiable core in S52SAT
Contribution: additional reasoning using selectors

Translation with selectors on conjunctions and disjunctions

\[
\begin{align*}
\text{tr}_{\text{sel}}(\phi, n) &= \text{tr}_{\text{sel}'}(\phi, 1, n) \\
\text{tr}_{\text{sel}'}(p, i, n) &= p_i \\
\text{tr}_{\text{sel}'}(\neg p, i, n) &= \neg p_i \\
\text{tr}_{\text{sel}'}(\psi \land \chi, i, n) &= \text{tr}_{\text{sel}'}(\psi, i, n) \land \text{tr}_{\text{sel}'}(\chi, i, n) \\
\text{tr}_{\text{sel}'}(\psi \lor \chi, i, n) &= \text{tr}_{\text{sel}'}(\psi, i, n) \lor \text{tr}_{\text{sel}'}(\chi, i, n) \\
\text{tr}_{\text{sel}'}(\Box \psi, i, n) &= \bigwedge_{j=1}^{n} (\neg s_j \lor (\text{tr}_{\text{sel}}'(\psi, j, n))) \\
\text{tr}_{\text{sel}'}(\Diamond \psi, i, n) &= \bigvee_{j=1}^{n} (s_j \land (\text{tr}_{\text{sel}}'(\psi, j, n)))
\end{align*}
\]
Contributions: Important lemmas for the approach

Lemma 1
If \( \text{tr}_\text{sel}(\phi, n) \) is unsatisfiable and the solver returns an unsatisfiable core of size \( m \), then all groups of \( m \) selectors are also an unsatisfiable core.

Lemma 2
If \( \text{tr}_\text{sel}(\phi, n) \) is unsatisfiable and the solver returns an unsatisfiable core of size \( m \) then \( \forall n' \in \{1, \ldots, (\text{dd}(\phi) - m)\} \), \( \text{tr}_{\text{sel}}(\phi, n') \) is unsatisfiable.
Contributions: S52SAT 1 to \( N_c \)

**Data:** \( \phi \), a modal logic formula

**Result:** \( \langle M, w \rangle \) such that \( \langle M, w \rangle \models_{min} \phi \), else UNSAT

```plaintext
1 begin
2    \( b \leftarrow 1 \); 
3    \( n \leftarrow \text{dd}(\phi) + 1 \); 
4    \( \langle r, s \rangle \leftarrow \text{glucose}(\text{tr}_{\text{sel}}(\phi, n), b) \); 
5    while (\( r \neq \text{SAT} \land (b \leq n) \)) do 
6        \( b \leftarrow b + (n - |s| + 1) \); 
7        \( \langle r, s \rangle \leftarrow \text{glucose}(\text{tr}_{\text{sel}}(\phi, n), b) \); 
8    end 
9    if \( r \neq \text{SAT} \) then 
10       return UNSAT ; 
11    else 
12       \( \langle M, w \rangle \leftarrow \text{getS5Model}(s) \); 
13       return \( \langle M, w \rangle \); 
14    end 
15 end
```
Experimental settings

Settings
- Cluster of Xeon 4 cores, 3.3 GHz, running CentOS 6.4
- Memory limit: 32 GB
- Time limit: 900 seconds (per solver per benchmark)

Benchmarks
- Random QBF (MQBF) formulae [MD00, KT13]
- Random 3CNF formulae of modal depths 1 and 2 [PS03]
- Crafted LWB K, KT and S4 formulae [BHS00]
Experimental settings

**Solvers**

- S52SAT [CLL+17] + 6 different strategies:
  - $1toN_c - 1toN$
  - $Nto1_c - Nto1$
  - $Dicho_c - Dicho$
- CNF with soft clauses with MaxSAT solvers:
  - maxHS-b [DB11]
  - mscg2015b [dRMIS14]
  - MSUnCore [HMMS11]
- MCS extraction with the LBX solver [MPM15]
## Experiments on 3CNF

<table>
<thead>
<tr>
<th>Method</th>
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<th>md=2</th>
<th>Total</th>
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<tr>
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<td>17 / 355</td>
<td>72 / 457</td>
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<td>Nto1</td>
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<td>0 / 457</td>
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<td>Dicho</td>
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<td>9 / 355</td>
<td>35 / 457</td>
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<td>62 / 102</td>
<td>27 / 355</td>
<td>89 / 457</td>
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<td>0 / 102</td>
<td>0 / 355</td>
<td>0 / 457</td>
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<tr>
<td>Dicho&lt;sub&gt;c&lt;/sub&gt;</td>
<td>40 / 102</td>
<td>17 / 355</td>
<td>57 / 457</td>
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<tr>
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<td>17 / 355</td>
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<td>17 / 355</td>
<td>57 / 457</td>
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<td>VBS</td>
<td>62 / 102</td>
<td>27 / 355</td>
<td>89 / 457</td>
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**Table:** #instances solved in 3CNF
### Experiments on MQBF

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<td>227 / 617</td>
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<tr>
<td>Dicho</td>
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<td>0 / 240</td>
<td>171 / 240</td>
<td>0 / 240</td>
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<tr>
<td>1toN&lt;sub&gt;c&lt;/sub&gt;</td>
<td>56  / 56</td>
<td>0 / 240</td>
<td>171 / 240</td>
<td>0 / 240</td>
<td>227 / 617</td>
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<td>0 / 240</td>
<td>0 / 240</td>
<td>0 / 240</td>
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<tr>
<td>Dicho&lt;sub&gt;c&lt;/sub&gt;</td>
<td>56  / 56</td>
<td>0 / 240</td>
<td>171 / 240</td>
<td>0 / 240</td>
<td>227 / 617</td>
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<tr>
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<td>0 / 240</td>
<td>0 / 240</td>
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<tr>
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<td>167 / 240</td>
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<td>0 / 240</td>
<td>171 / 240</td>
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<td>227 / 617</td>
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**Table:** #instances solved in MQBF
## Results on LWB

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<td>160 / 504</td>
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<td>365 / 1512</td>
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<tr>
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<td>21 / 504</td>
<td>42 / 504</td>
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<tr>
<td>LBX</td>
<td>118 / 504</td>
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</tr>
</tbody>
</table>

**Table:** #instances solved in LWB
Impact of the additional reasoning with selectors (LWB)
Comparison of the number of calls to the SAT oracle
Conclusion & Perspectives

Conclusion

- We defined the problem MinS5-SAT
- Different techniques to solve it
- An elegant way to use selectors to minimize a model
- Empirically, formulae in different modal logics are sometimes S5-satisfiable

Perspectives

- Use S5\_2SAT as preprocessor for modal logic K
- Try to minimize Kripke model for other modal logics
- Find practical applications that can be translated into modal logic S5
Conclusion
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