On Division Versus Saturation in Pseudo-Boolean Solving

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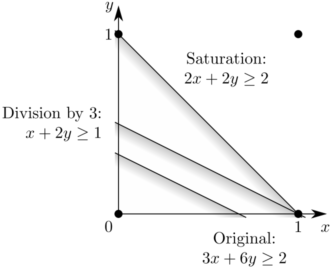

The Problem

- pseudo-Boolean (PB) constraints, i.e. $\{0, 1\}$-linear inequalities
- use $\bar{x} = (1 - x)$, allows us to have no negative coefficients

Example:

$$\begin{aligned} h_1 + h_2 + \bar{x}_1 + \bar{x}_2 &\geq 1 \\ \bar{h}_1 + \bar{x}_2 + \bar{x}_3 &\geq 2 \\ \bar{h}_2 + \bar{x}_1 + \bar{x}_3 &\geq 2 \\ x_1 + x_2 + x_3 &\geq 2 \end{aligned}$$

Want to answer:

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- How hard is it to show that there is no solution? It depends...

Solving Pseudo-Boolean Problems

- NP-hard $\Rightarrow$ can't expect efficient solution in general
- there are multiple approaches for solving PB problems
- our work focuses on PB solvers, i.e., algorithms...
  - similar to conflict-driven clause learning (CDCL) SAT solvers
  - using PB constraints to analyse conflicts
  - in practice worse than known theoretic limitations
- goal: understand power of reasoning

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This work:

- study so called saturation rule and division rule
- as used in PB solvers
- show that they are incomparable

Cutting Planes in PB Solvers

Literal Axioms

$$\frac{}{x \geq 0} \quad \frac{}{\bar{x} \geq 0}$$

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Generalized Resolution

(positive linear combination eliminating variable)

$$\frac{h_1 + h_2 + \bar{x}_1 + \bar{x}_2 \geq 1 \qquad \bar{h}_1 + \bar{x}_1 + \bar{x}_3 \geq 2}{\underbrace{h_1 + \bar{h}_1}_{=h_1+1-h_1} + h_2 + 2\bar{x}_1 + \bar{x}_2 + \bar{x}_3 \geq 3}$$

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Cutting Planes in PB Solvers — Boolean Rule

Division (divide and round up coefficients and right hand side)

used in [EN18]

$$\frac{x_1 + 2x_2 + 2x_3 \geq 3}{x_1 + x_2 + x_3 \geq 2} \text{ Divide by 2}$$

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or

Saturation (reduce to min of coefficient and right hand side)
used in [DG02, CK05, SS06, LP10]

$$\frac{6x + 3y + z_1 + z_2 \geq 3}{3x + 3y + z_1 + z_2 \geq 3}$$

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How do these rules compare?

- Is one of them strictly better?
- Or are they incomparable?

Cutting Planes — Example

$$\dfrac{\dfrac{\dfrac{\dfrac{h_1 + h_2 + \bar{x}_1 + \bar{x}_2 \geq 1 \qquad \bar{h}_1 + \bar{x}_1 + \bar{x}_3 \geq 2}{} \qquad \bar{h}_2 + \bar{x}_2 + \bar{x}_3 \geq 2}{}}{} \qquad x_1 + x_2 + x_3 \geq 2}{}$$

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$$\cfrac{\cfrac{\cfrac{\cfrac{h_1 + h_2 + \bar{x}_1 + \bar{x}_2 \geq 1 \qquad \bar{h}_1 + \bar{x}_1 + \bar{x}_3 \geq 2}{h_2 + 2\bar{x}_1 + \bar{x}_2 + \bar{x}_3 \geq 2} \qquad \bar{h}_2 + \bar{x}_2 + \bar{x}_3 \geq 2}{2\bar{x}_1 + 2\bar{x}_2 + 2\bar{x}_3 \geq 3}}{} \qquad x_1 + x_2 + x_3 \geq 2}{}$$

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Is division stronger than saturation?

- generalized resolution can derive (4)
- division can derive (5)
- saturation does not change (4)

$$h_1 + h_2 + \bar{x}_1 + \bar{x}_2 \geq 1 \quad (1)$$

$$\bar{h}_1 + \bar{x}_2 + \bar{x}_3 \geq 2 \quad (2)$$

$$\bar{h}_2 + \bar{x}_1 + \bar{x}_3 \geq 2 \quad (3)$$

$$2\bar{x}_1 + 2\bar{x}_2 + 2\bar{x}_3 \geq 3 \quad (4)$$

$$\bar{x}_1 + \bar{x}_2 + \bar{x}_3 \geq 2 \quad (5)$$

Is division stronger than saturation?

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h_1 + h_2 + \bar{x}_1 + \bar{x}_2 & & \geq 1 & (1) \\
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\end{array}
$$

there are formulas that...

- contain constraints similar to (1)-(3)
- are unsatisfiable
- showing unsatisfiability using generalized resolution and...
  - saturation requires an exponential number of steps
  - division can be done in a linear number of steps

Is division stronger than saturation? As in [VEGC+18].

$$
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there are formulas that...

- contain constraints similar to (1)-(3)
- are unsatisfiable
- showing unsatisfiability using ~~generalized resolution~~ arbitrary positive linear combination and...
  - saturation requires an exponential number of steps
  - division can be done in a linear number of steps

Difference to [VEGC+18]

- [VEGC+18] does not apply to generalized resolution
- problem: PB solver do use generalized resolution
  $\Rightarrow$ used formula [MN14] is always hard for PB solver
  (no matter if saturation or division is used)

Difference to [VEGC$^+$18]

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- problem: PB solver do use generalized resolution
  $\Rightarrow$ used formula [MN14] is always hard for PB solver
  (no matter if saturation or division is used)

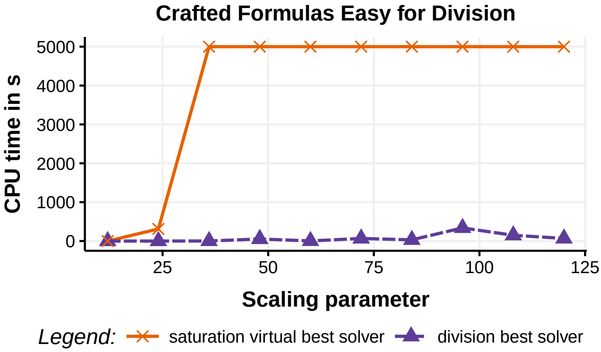
- we modify formula to allow generalized resolution
  (via helper variables $h_1, h_2, \dots$)
- we show that generalized resolution and...
  - saturation still requires an exponential number of steps
  - division can now derive UNSAT in a linear number of steps

Practical Experiments: Division Stronger Than Saturation

- saturation based solvers are guaranteed to run slow
- can division based solvers show unsatisfiability fast?

Practical Experiments: Division Stronger Than Saturation

- saturation based solvers are guaranteed to run slow
- can division based solvers show unsatisfiability fast?
  - yes, but sensitive to other settings



Is saturation stronger than division?

$$Rx + Ry + \sum_{i=1}^{R} z_i \geq R \quad (6)$$

$$Rx + R\bar{y} + \sum_{i=R+1}^{2R} z_i \geq R \quad (7)$$

$$\overline{\qquad\qquad\qquad\qquad\qquad\qquad}$$

$$2Rx + \sum_{i=1}^{2R} z_i \geq R \quad (8)$$

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$$Rx + \sum_{i=1}^{2R} z_i \geq R \quad (9)$$

- generalized resolution can derive (8)
- saturation can derive (9) in one step

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- generalized resolution can derive (8)
- saturation can derive (9) in one step
- division can derive (9), but requires at least $\sqrt{R}$ steps

Proof Sketch: Define Suitable Potential Function

$$\mathcal{P}(ax + by + b'\bar{y} + \sum c_i z_i \geq A) := \ln\left((2a + b + b')/A\right)$$

Examples:

$$\mathcal{P}(C_{\text{start}}) := \quad \mathcal{P}(Rx + Ry + \sum_{i=1}^{R} z_i \geq R) = \ln\left(3R/R\right) = \ln\left(3\right)$$

$$\mathcal{P}(C_{\text{end}}) := \quad \mathcal{P}(Rx + \sum_{i=1}^{2R} z_i \geq R) = \ln\left(2R/R\right) = \ln\left(2\right)$$

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Important properties:

- needs to change:
  $\mathcal{P}(C_{\text{start}}) - \mathcal{P}(C_{\text{end}}) \geq 1/6$
- doesn't change with generalized resolution:
  $\mathcal{P}(C_1 \oplus C_2) \geq \min\{\mathcal{P}(C_1), \mathcal{P}(C_2)\}$
- division only changes $\mathcal{P}$ by a small amount:
  $\mathcal{P}(C/k) \geq \mathcal{P}(C) - 1/\sqrt{R}$

Conclusion

- division can be provably stronger than saturation
- saturation can be provably stronger than division
  (for deriving specific constraint)

Future Research Directions

- division rule and saturation rule seem incomparable
  $\Rightarrow$ implement adaptive choice between division and saturation
- practical results sensitive to other settings
  $\Rightarrow$ better understanding of implementation choices desirable
- for some problems mixed integer programming is more efficient
  $\Rightarrow$ try to use the best from both worlds

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Thank you for your attention!

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