In Between Resolution and Cutting Planes
Proof Systems for Pseudo-Boolean SAT Solving

Marc Vinyals

KTH Royal Institute of Technology
Stockholm, Sweden

Joint work with Jan Elffers, Jesús Giráldez-Cru, Stephan Gocht, and Jakob Nordström

Pragmatics of Constraint Reasoning Workshop
August 28 2017, Melbourne, Australia

What do we do

Study pseudo-Boolean solvers from proof complexity point of view

Question

How powerful are pseudo-Boolean solvers?

Build two kinds of formulas

- solvers can perform well with good heuristics
- solvers do not exploit power of pseudo-Boolean constraints

The CDCL Algorithm

```
while not solved :
  unit propagate
  if conflict :
    learn
    backtrack
  else :
    decide variable
```

$x \vee y \quad x \vee \overline{y} \vee z \quad x \vee \overline{y} \vee \overline{z} \quad \overline{x} \vee y \quad \overline{x} \vee \overline{y}$

Database

Assignment

The CDCL Algorithm

```
while not solved :
  unit propagate
  if conflict :
    learn
    backtrack
  else :
    decide variable
```

$x \vee y \quad x \vee \overline{y} \vee z \quad x \vee \overline{y} \vee \overline{z} \quad \overline{x} \vee y \quad \overline{x} \vee \overline{y}$

Database

Assignment

The CDCL Algorithm

```
while not solved :
  unit propagate
  if conflict :
    learn
    backtrack
  else :
    decide variable
```

$x \vee y \quad x \vee \overline{y} \vee z \quad x \vee \overline{y} \vee \overline{z} \quad \overline{x} \vee y \quad \overline{x} \vee \overline{y}$

Database

Assignment

$x \stackrel{\mathsf{d}}{=} 0$

The CDCL Algorithm

```
while not solved :
  unit propagate
  if conflict :
    learn
    backtrack
  else :
    decide variable
```

$x \lor y \quad x \lor \overline{y} \lor z \quad x \lor \overline{y} \lor \overline{z} \quad \overline{x} \lor y \quad \overline{x} \lor \overline{y}$

Database

Assignment

$x \overset{\mathsf{d}}{=} 0 \quad y \overset{x \lor y}{=} 1$

The CDCL Algorithm

```
while not solved :
  unit propagate
  if conflict :
    learn
    backtrack
  else :
    decide variable
```

$x \vee y \quad x \vee \overline{y} \vee z \quad x \vee \overline{y} \vee \overline{z} \quad \overline{x} \vee y \quad \overline{x} \vee \overline{y}$

Database

Assignment

$x \stackrel{\mathsf{d}}{=} 0 \quad y \stackrel{x \vee y}{=} 1 \quad z \stackrel{x \vee \overline{y} \vee z}{=} 1$

The CDCL Algorithm

```
while not solved :
  unit propagate
  if conflict :
    learn
    backtrack
  else :
    decide variable
```

$x \lor y \quad x \lor \overline{y} \lor z \quad x \lor \overline{y} \lor \overline{z} \quad \overline{x} \lor y \quad \overline{x} \lor \overline{y}$

Database

Assignment

$x \overset{\mathsf{d}}{=} 0 \quad y \overset{x \lor y}{=} 1 \quad z \overset{x \lor \overline{y} \lor z}{=} 1$

The CDCL Algorithm

```
while not solved :
  unit propagate
  if conflict :
    learn
    backtrack
  else :
    decide variable
```

$x \vee y \quad x \vee \overline{y} \vee z \quad x \vee \overline{y} \vee \overline{z} \quad \overline{x} \vee y \quad \overline{x} \vee \overline{y}$

Database

x

Assignment

$x \stackrel{\mathsf{d}}{=} 0 \quad y \stackrel{x \vee y}{=} 1 \quad z \stackrel{x \vee \overline{y} \vee z}{=} 1$

The CDCL Algorithm

```
while not solved :
  unit propagate
  if conflict :
    learn
    backtrack
  else :
    decide variable
```

$x \vee y \quad x \vee \overline{y} \vee z \quad x \vee \overline{y} \vee \overline{z} \quad \overline{x} \vee y \quad \overline{x} \vee \overline{y}$

Database

x

Assignment

The CDCL Algorithm

```
while not solved :
  unit propagate
  if conflict :
    learn
    backtrack
  else :
    decide variable
```

$x \lor y \quad x \lor \overline{y} \lor z \quad x \lor \overline{y} \lor \overline{z} \quad \overline{x} \lor y \quad \overline{x} \lor \overline{y}$

Database

x

Assignment

$x \stackrel{x}{=} 1$

The CDCL Algorithm

```
while not solved :
  unit propagate
  if conflict :
    learn
    backtrack
  else :
    decide variable
```

$x \vee y \quad x \vee \overline{y} \vee z \quad x \vee \overline{y} \vee \overline{z} \quad \overline{x} \vee y \quad \overline{x} \vee \overline{y}$

Database

x

Assignment

$x \overset{x}{=} 1 \quad y \overset{\overline{x} \vee y}{=} 1$

The CDCL Algorithm

```
while not solved :
  unit propagate
  if conflict :
    learn
    backtrack
  else :
    decide variable
```

$x \vee y \quad x \vee \overline{y} \vee z \quad x \vee \overline{y} \vee \overline{z} \quad \overline{x} \vee y \quad \overline{x} \vee \overline{y}$

Database

x

Assignment

$x \stackrel{x}{=} 1 \quad y \stackrel{\overline{x} \vee y}{=} 1$

The CDCL Algorithm

```
while not solved :
  unit propagate
  if conflict :
    learn
    backtrack
  else :
    decide variable
```

$x \vee y \quad x \vee \overline{y} \vee z \quad x \vee \overline{y} \vee \overline{z} \quad \overline{x} \vee y \quad \overline{x} \vee \overline{y}$

Database

$x \quad \perp$

Assignment

$x \overset{x}{=} 1 \quad y \overset{\overline{x} \vee y}{=} 1$

Conflict Analysis

- Say there is a conflict with variable z

$x \lor y \quad x \lor \overline{y} \lor z \quad x \lor \overline{y} \lor \overline{z}$

Assignment ρ

$x \overset{\mathsf{d}}{=} 0 \quad y \overset{x \lor y}{=} 1 \quad z \overset{x \lor \overline{y} \lor z}{=} 1$

Conflict Analysis

- Say there is a conflict with variable z
- Some clause $C \vee \overline{z}$ caused the conflict

$x \vee y \quad x \vee \overline{y} \vee z \quad x \vee \overline{y} \vee \overline{z}$

Assignment ρ

$x \overset{\mathsf{d}}{=} 0 \quad y \overset{x \vee y}{=} 1 \quad z \overset{x \vee \overline{y} \vee z}{=} 1$

Conflict Analysis

- Say there is a conflict with variable z
- Some clause $C \vee \overline{z}$ caused the conflict
- Another clause $D \vee z$ propagated z

$x \vee y \quad x \vee \overline{y} \vee z \quad x \vee \overline{y} \vee \overline{z}$

Assignment ρ

$x \overset{\mathsf{d}}{=} 0 \quad y \overset{x \vee y}{=} 1 \quad z \overset{x \vee \overline{y} \vee z}{=} 1$

Conflict Analysis

- Say there is a conflict with variable z
- Some clause $C \lor \bar{z}$ caused the conflict
- Another clause $D \lor z$ propagated z
- Use resolution rule to derive $C \lor D$.

$x \lor y \quad x \lor \bar{y} \lor z \quad x \lor \bar{y} \lor \bar{z}$

Assignment ρ

$x \overset{\mathsf{d}}{=} 0 \quad y \overset{x \lor y}{=} 1 \quad z \overset{x \lor \bar{y} \lor z}{=} 1$

Resolution

$$\frac{x \lor \bar{y} \lor z \qquad x \lor \bar{y} \lor \bar{z}}{x \lor \bar{y}}$$

Conflict Analysis

- Say there is a conflict with variable z
- Some clause $C \vee \overline{z}$ caused the conflict
- Another clause $D \vee z$ propagated z
- Use resolution rule to derive $C \vee D$.
- Remove z from assignment.

$x \vee y \quad x \vee \overline{y} \vee z \quad x \vee \overline{y} \vee \overline{z}$

Assignment $\rho \setminus \{z\}$

$x \stackrel{\mathsf{d}}{=} 0 \quad y \stackrel{x \vee y}{=} 1$

Resolution

$$\frac{x \vee \overline{y} \vee z \qquad x \vee \overline{y} \vee \overline{z}}{x \vee \overline{y}}$$

Conflict Analysis

- Say there is a conflict with variable z
- Some clause $C \vee \overline{z}$ caused the conflict
- Another clause $D \vee z$ propagated z
- Use resolution rule to derive $C \vee D$.
- Remove z from assignment.
- ρ falsifies C, ρ falsifies $D \Rightarrow$
  $\rho \setminus \{z\}$ falsifies $C \vee D$.

$x \vee y \quad x \vee \overline{y} \vee z \quad x \vee \overline{y} \vee \overline{z}$

Assignment $\rho \setminus \{z\}$

$x \overset{\mathsf{d}}{=} 0 \quad y \overset{x \vee y}{=} 1$

Resolution

$$\frac{x \vee \overline{y} \vee z \qquad x \vee \overline{y} \vee \overline{z}}{x \vee \overline{y}}$$

Conflict Analysis

- Say there is a conflict with variable z
- Some clause $C \vee \overline{z}$ caused the conflict
- Another clause $D \vee z$ propagated z
- Use resolution rule to derive $C \vee D$.
- Remove z from assignment.
- ρ falsifies C, ρ falsifies $D \Rightarrow$ $\rho \setminus \{z\}$ falsifies $C \vee D$.
- Repeat until there is no reason for propagation.

$x \vee y \quad x \vee \overline{y} \vee z \quad x \vee \overline{y} \vee \overline{z}$

Assignment $\rho \setminus \{z\}$

$x \overset{\mathsf{d}}{=} 0 \quad y \overset{x \vee y}{=} 1$

Resolution

$$\frac{x \vee \overline{y} \vee z \qquad x \vee \overline{y} \vee \overline{z}}{x \vee \overline{y}}$$

The Power of CDCL Solvers

All CDCL proofs are resolution proofs

Lower bound for resolution length $\Rightarrow$ lower bound for CDCL run time

*(Ignoring preprocessing)

The Power of CDCL Solvers

All CDCL proofs are resolution proofs
Lower bound for resolution length $\Rightarrow$ lower bound for CDCL run time

*(Ignoring preprocessing)

And the opposite direction?

Theorem [Pipatsrisawat, Darwiche '09; Atserias, Fichte, Thurley '09]

CDCL $\equiv$ Resolution

- CDCL can simulate any resolution proof
- Assumes optimal decision and erasure heuristics

More Powerful Solvers

Resolution is a weak proof system

- e.g. cannot count
- $x_1 + \cdots + x_n = n/2$ needs exponentially many clauses

More Powerful Solvers

Resolution is a weak proof system

- e.g. cannot count
- $x_1 + \cdots + x_n = n/2$ needs exponentially many clauses

Pseudo-Boolean constraints more expressive

$$x_1 + \cdots + x_n \geq n/2$$
$$\overline{x_1} + \cdots + \overline{x_n} \geq n/2$$

Build solvers with pseudo-Boolean constraints?

Pseudo-Boolean CDCL

CDCL with pseudo-Boolean constraints is tricky

Pseudo-Boolean CDCL

CDCL with pseudo-Boolean constraints is tricky

- Several variables can propagate in one go

$$2x + y + z \geq 2$$

Assignment

$$x \stackrel{\mathsf{d}}{=} 0$$

Pseudo-Boolean CDCL

CDCL with pseudo-Boolean constraints is tricky

- Several variables can propagate in one go

$$2x + y + z \geq 2$$

Assignment

$$x \stackrel{\mathsf{d}}{=} 0 \quad y \stackrel{2x+y+z\geq 2}{=} 1 \quad z \stackrel{2x+y+z\geq 2}{=} 1$$

Pseudo-Boolean CDCL

CDCL with pseudo-Boolean constraints is tricky

- Several variables can propagate in one go
- Derived constraint not always falsified by assignment

$$x_1 + 2\overline{x_3} + x_4 + 2x_6 \geq 2 \qquad x_2 + x_5 + 2\overline{x_6} \geq 2$$

Assignment

$$x_1 \stackrel{\mathsf{d}}{=} 0 \quad x_2 \stackrel{\mathsf{d}}{=} 0 \quad x_3 \stackrel{\mathsf{d}}{=} 1$$

Database

Pseudo-Boolean CDCL

CDCL with pseudo-Boolean constraints is tricky

- Several variables can propagate in one go
- Derived constraint not always falsified by assignment

$x_1 + 2\overline{x_3} + x_4 + 2x_6 \geq 2 \qquad x_2 + x_5 + 2\overline{x_6} \geq 2$

Assignment

$x_1 \overset{\mathsf{d}}{=} 0 \quad x_2 \overset{\mathsf{d}}{=} 0 \quad x_3 \overset{\mathsf{d}}{=} 1 \quad x_6 \overset{x_1+2\overline{x_3}+x_4+2x_6 \geq 2}{=} 1$

Database

Pseudo-Boolean CDCL

CDCL with pseudo-Boolean constraints is tricky

- Several variables can propagate in one go
- Derived constraint not always falsified by assignment

$$x_1 + 2\overline{x_3} + x_4 + 2x_6 \geq 2 \qquad x_2 + x_5 + 2\overline{x_6} \geq 2$$

Assignment

$$x_1 \stackrel{\mathsf{d}}{=} 0 \quad x_2 \stackrel{\mathsf{d}}{=} 0 \quad x_3 \stackrel{\mathsf{d}}{=} 1$$

Database

$$x_1 + x_2 + 2\overline{x_3} + x_4 + x_5 \geq 2$$

Pseudo-Boolean CDCL

CDCL with pseudo-Boolean constraints is tricky

- Several variables can propagate in one go
- Derived constraint not always falsified by assignment

Yet all of this can be fixed

Cutting Planes

All pseudo-Boolean proofs are cutting planes proofs

Cutting Planes

All pseudo-Boolean proofs are cutting planes proofs

Work with linear pseudo-Boolean inequalities

$$x \vee \overline{y} \quad \rightarrow \quad x + \overline{y} \geq 1 \quad \equiv \quad x + (1 - y) \geq 1$$

Cutting Planes

All pseudo-Boolean proofs are cutting planes proofs

Work with linear pseudo-Boolean inequalities

$$x \vee \overline{y} \quad \rightarrow \quad x + \overline{y} \geq 1 \quad \equiv \quad x + (1 - y) \geq 1$$

Rules

Variable axioms

$$\frac{}{x \geq 0} \qquad \frac{}{-x \geq -1}$$

Addition

$$\frac{\sum a_i x_i \geq a \qquad \sum b_i x_i \geq b}{\sum (\alpha a_i + \beta b_i) x_i \geq \alpha a + \beta b}$$

Division

$$\frac{\sum a_i x_i \geq a}{\sum (a_i / k) x_i \geq \lceil a / k \rceil}$$

Cutting Planes

All pseudo-Boolean proofs are cutting planes proofs

Work with linear pseudo-Boolean inequalities

$$x \vee \overline{y} \quad \to \quad x + \overline{y} \geq 1 \quad \equiv \quad x + (1 - y) \geq 1$$

Rules

Variable axioms

$$\frac{}{x \geq 0} \qquad \frac{}{-x \geq -1}$$

Addition

$$\frac{\sum a_i x_i \geq a \qquad \sum b_i x_i \geq b}{\sum (\alpha a_i + \beta b_i) x_i \geq \alpha a + \beta b}$$

Division

$$\frac{\sum a_i x_i \geq a}{\sum (a_i / k) x_i \geq \lceil a/k \rceil}$$

Goal: derive $0 \geq 1$

Addition in Practice

Addition

$$\frac{\sum a_i x_i \geq a \qquad \sum b_i x_i \geq b}{\sum(\alpha a_i + \beta b_i) x_i \geq \alpha a + \beta b}$$

- Unbounded choices
- Need a reason to add inequalities:
  - One conflicting variable
  - Conflict disappears after addition

Addition in Practice

Addition

$$\frac{\sum a_i x_i \geq a \qquad \sum b_i x_i \geq b}{\sum(\alpha a_i + \beta b_i) x_i \geq \alpha a + \beta b}$$

- Unbounded choices
- Need a reason to add inequalities:
  - One conflicting variable
  - Conflict disappears after addition

Cancelling Addition

Some variable cancels: $\alpha a_i + \beta b_i = 0$

Division in Practice

Division

$$\frac{\sum a_i x_i \geq a}{\sum (a_i/k) x_i \geq \lceil a/k \rceil}$$

- Too expensive

Division in Practice

Division

$$\frac{\sum a_i x_i \geq a}{\sum (a_i/k) x_i \geq \lceil a/k \rceil}$$

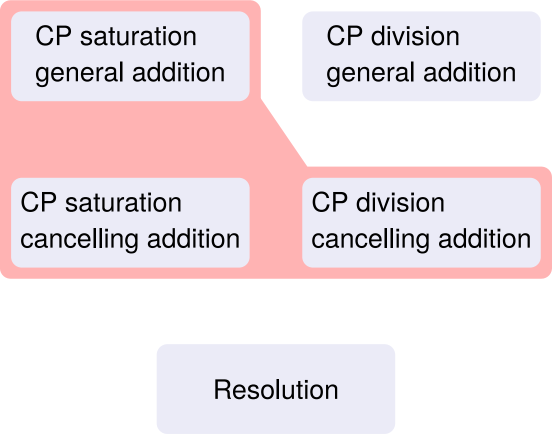
- Too expensive

Saturation

$$\frac{\sum a_i x_i \geq a}{\sum \min(a, a_i) x_i \geq a}$$

Proof Systems

Power of subsystems of CP?



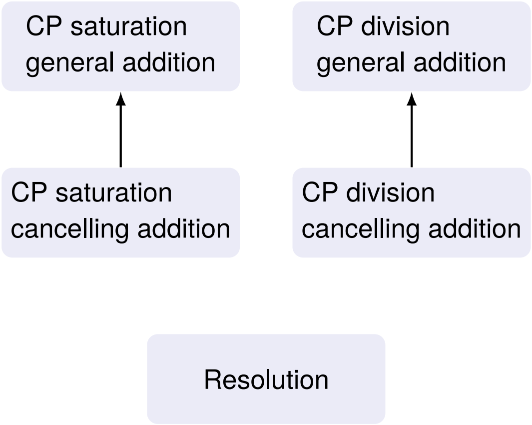
Results

Theorem

On CNF inputs all subsystems as weak as resolution

- No subsystem is implicationally complete
- Solver becomes very sensitive to the encoding

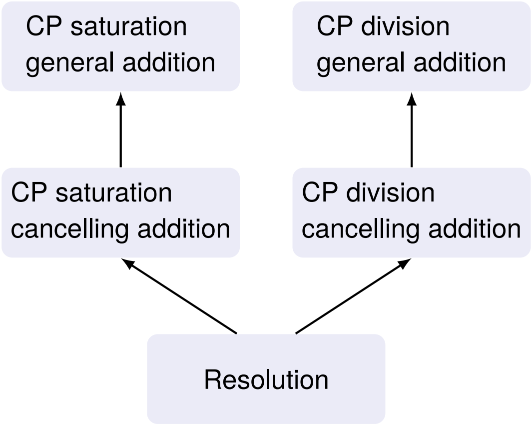
Proof Systems



Cancelling addition is a particular case of addition

$A \longrightarrow B$: B simulates A (with only polynomial loss)

Proof Systems

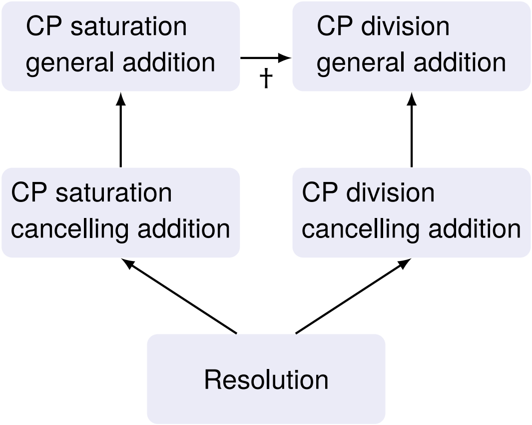


$A \longrightarrow B$: B simulates A (with only polynomial loss)

All subsystems simulate resolution

- Trivial over CNF inputs
- Also holds over linear pseudo-Boolean inputs

Proof Systems



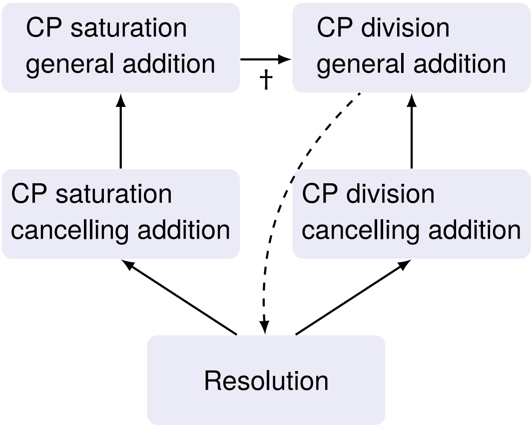
Repeated divisions simulate saturation

- Polynomial simulation only if polynomial coefficients

$A \longrightarrow B$: B simulates A (with only polynomial loss)

†: known only for polynomial-size coefficients

Proof Systems



CP stronger than resolution

- Pigeonhole principle
- Subset cardinality

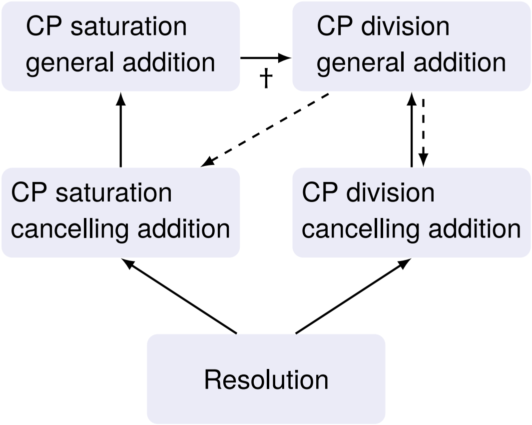
have proofs of size

- polynomial in CP
- exponential in resolution

$A \longrightarrow B$: B simulates A (with only polynomial loss)
$A \dashrightarrow B$: B cannot simulate A (separation)

†: known only for polynomial-size coefficients

Proof Systems



Cancellation $\equiv$ Resolution

- Over CNF inputs

[Hooker '88]

- Pigeonhole principle
- Subset cardinality

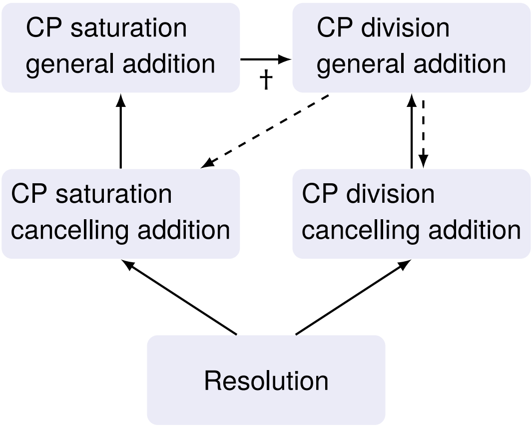
have proofs of size

- polynomial in CP
- exponential in resolution

$A \longrightarrow B$: B simulates A (with only polynomial loss)
$A \dashrightarrow B$: B cannot simulate A (separation)

†: known only for polynomial-size coefficients

Proof Systems



$A \longrightarrow B$: B simulates A (with only polynomial loss)
$A \dashrightarrow B$: B cannot simulate A (separation)

†: known only for polynomial-size coefficients

Cancellation $\equiv$ Resolution

- Over CNF inputs

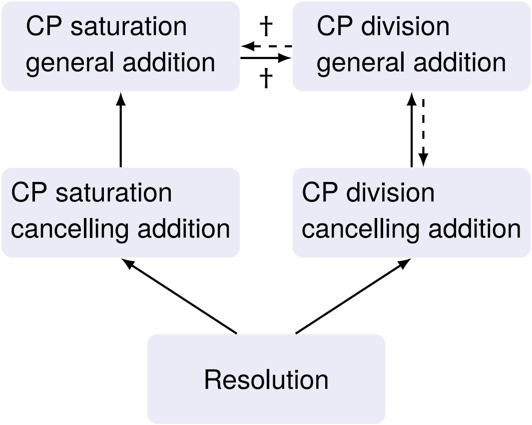
[Hooker '88]

- Pigeonhole principle
- Subset cardinality

have proofs of size

- polynomial in CP
- exponential in CP with cancelling addition and any rounding

Proof Systems



Saturation $\equiv$ Resolution

- Over CNF inputs

- Pigeonhole principle
- Subset cardinality

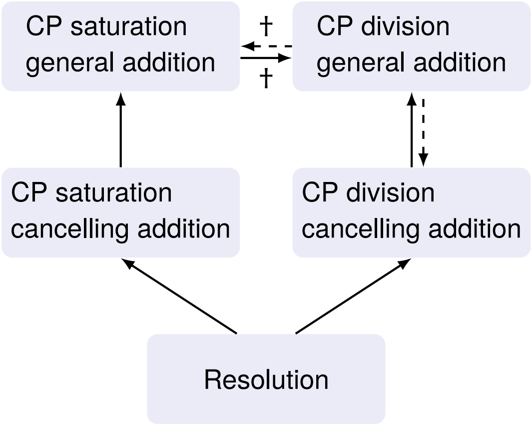
have proofs of size

- polynomial in CP
- exponential in resolution

$A \longrightarrow B$: B simulates A (with only polynomial loss)
$A \dashrightarrow B$: B cannot simulate A (separation)

†: known only for polynomial-size coefficients

Proof Systems



Saturation $\equiv$ Resolution

- Over CNF inputs

- Pigeonhole principle
- Subset cardinality

have proofs of size

- polynomial in CP
- exponential in CP with general addition and saturation

$A \longrightarrow B$: B simulates A (with only polynomial loss)
$A \dashrightarrow B$: B cannot simulate A (separation)

†: known only for polynomial-size coefficients

Easy Formulas

Pseudo-Boolean solvers $\equiv$ CP? No

Question

PB solvers $\equiv$ CP with cancelling addition and saturation?

Easy Formulas

Pseudo-Boolean solvers $\equiv$ CP? No

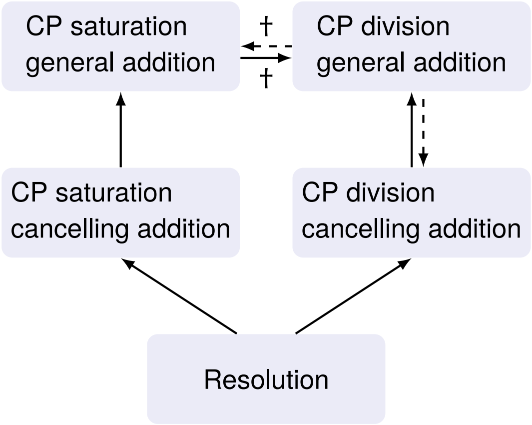
Question

PB solvers $\equiv$ CP with cancelling addition and saturation?

Craft combinatorial formulas easy for CP with cancelling addition and saturation

- All formulas without rational solutions
- Easy versions of NP-hard problems

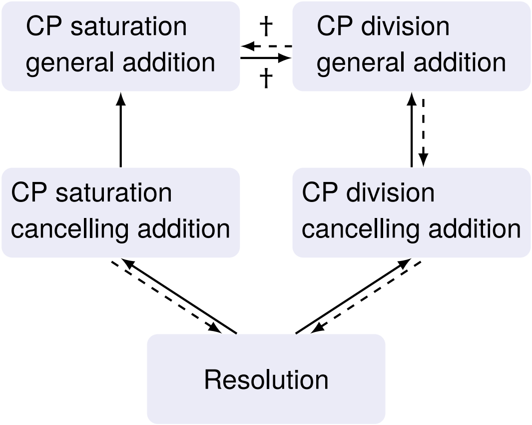
Proof Systems



$A \longrightarrow B$: B simulates A (with only polynomial loss)
$A \dashrightarrow B$: B cannot simulate A (separation)

†: known only for polynomial-size coefficients

Proof Systems



Pseudo-Boolean versions of

- Pigeonhole principle
- Subset cardinality
- …

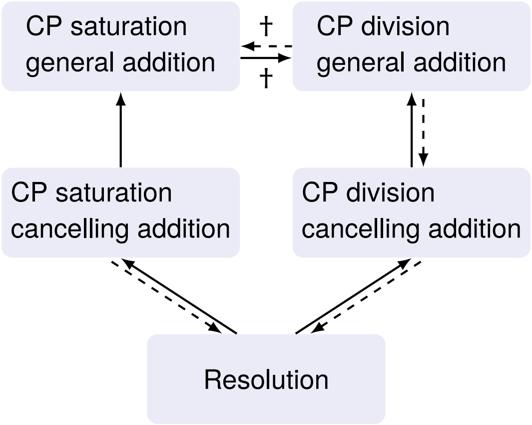
have proof of size

- polynomial in all CP subsystems
- exponential in resolution

$A \longrightarrow B$: B simulates A (with only polynomial loss)
$A \dashrightarrow B$: B cannot simulate A (separation)

†: known only for polynomial-size coefficients

Proof Systems



$A \longrightarrow B$: B simulates A (with only polynomial loss)
$A \dashrightarrow B$: B cannot simulate A (separation)

†: known only for polynomial-size coefficients

Pseudo-Boolean versions of

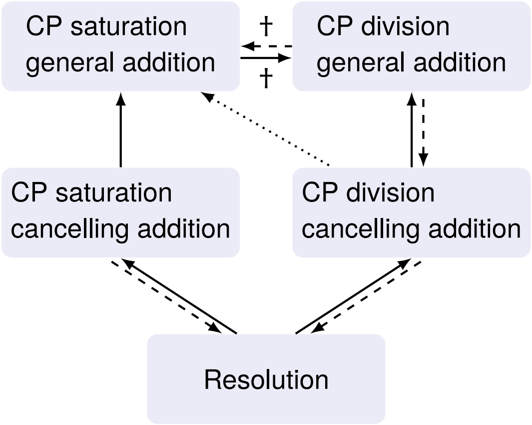
- Pigeonhole principle
- Subset cardinality
- ...

have proof of size

- polynomial in all CP subsystems
- exponential in resolution

CNF version exponential $\Rightarrow$
Cannot recover encoding $\Rightarrow$
Subsystems are incomplete

Proof Systems



Separation candidates
Some formulas have proof of size

- polynomial in CP with cancelling addition and division
- unknown in CP with general addition and saturation

$A \longrightarrow B$: B simulates A (with only polynomial loss)
$A \dashrightarrow B$: B cannot simulate A (separation)
$A \cdots\!\rightarrow B$: candidate for a separation
†: known only for polynomial-size coefficients

Take Home

Bad News

- On CNF inputs subsystems of CP $\equiv$ resolution
- Subsystems of CP implicationally incomplete

Take Home

Bad News

- On CNF inputs subsystems of CP $\equiv$ resolution
- Subsystems of CP implicationally incomplete

Good News

- Many formulas where PB solvers can shine
- Do PB solvers shine in practice?

Take Home

Bad News

- On CNF inputs subsystems of CP $\equiv$ resolution
- Subsystems of CP implicationally incomplete

Good News

- Many formulas where PB solvers can shine
- Do PB solvers shine in practice? (Stay tuned...)

Take Home

Bad News

- On CNF inputs subsystems of CP $\equiv$ resolution
- Subsystems of CP implicationally incomplete

Good News

- Many formulas where PB solvers can shine
- Do PB solvers shine in practice? (Stay tuned...)

Thanks!